A Sampling Method Based on LDPC Codes

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Introducton

Wish list for an ideal sampling method:

- Tractable computational costs
- Correctness
- Independence
- Good scaling behaviour
- Efficient use of randomness
-….

Cannot achieve all at the same time!

Our Contribution: a novel sampling method based on LDPC with **good scaling behaviour**, **good control over independence properties** and **makes efficient utilization of randomness**.

Proposed Sampling Scheme

Inspiration 1: Typical Sequences

Given $P_{\boldsymbol{x}}(\cdot)$ over $\{0,1\}^N$, $H_0 = -\frac{1}{N}\sum_{\boldsymbol{x}} P_{\boldsymbol{x}}(\boldsymbol{x}) \log_2 P_{\boldsymbol{x}}(\boldsymbol{x})$, set of typical sequences can be defined for $\forall t \in \mathbb{N}$

$$\mathcal{T}_\epsilon^t = \{(x^1, x^2, \dots, x^t) \in \{0,1\}^{Nt}: \left|\frac{1}{t}\sum_{i=1}^t P_{\boldsymbol{x}}(x^i) + NH_0\right| \le \epsilon\}$$

Asymptotic Equipartition Property: typical sequences are roughly equi-probable and dominate the probability!

Idea: typical sequences ⇒ samples (e.g. by marginalization).
But how to generate typical sequences?

Inspiration 2: Compression

Compression (of discrete source): **typical sequence ⇒ index**
We want: **index ⇒ typical sequence**
LDPC codes have been used for compression

(binary) LDPC Codes:

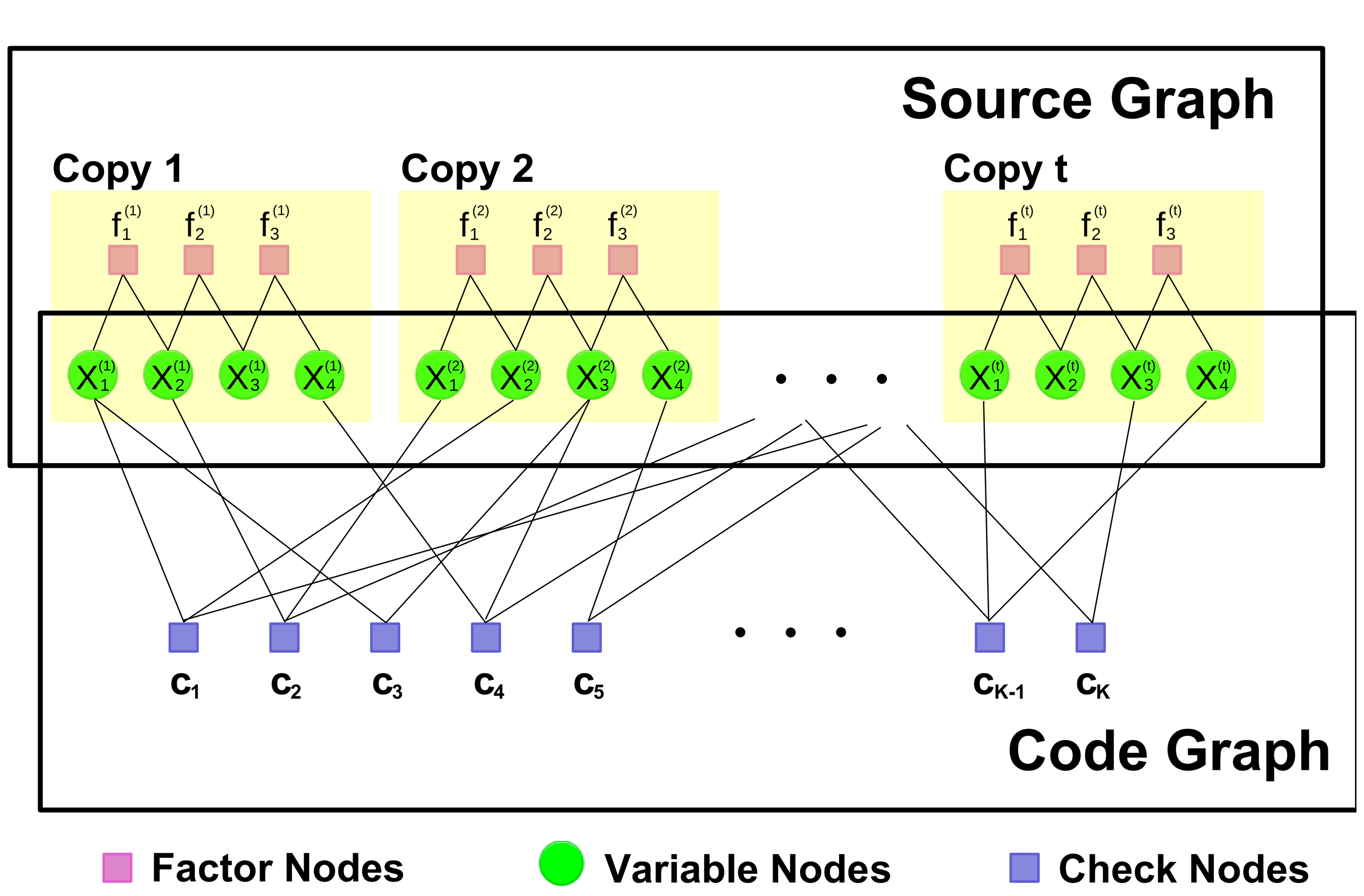
$$\mathbb{H} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 0 \end{pmatrix} \in \{0,1\}^{K \times Nt} \qquad \boldsymbol{z} = \mathbb{H}\begin{pmatrix} \boldsymbol{x}^1 \\ \vdots \\ \boldsymbol{x}^t \end{pmatrix} \pmod 2$$

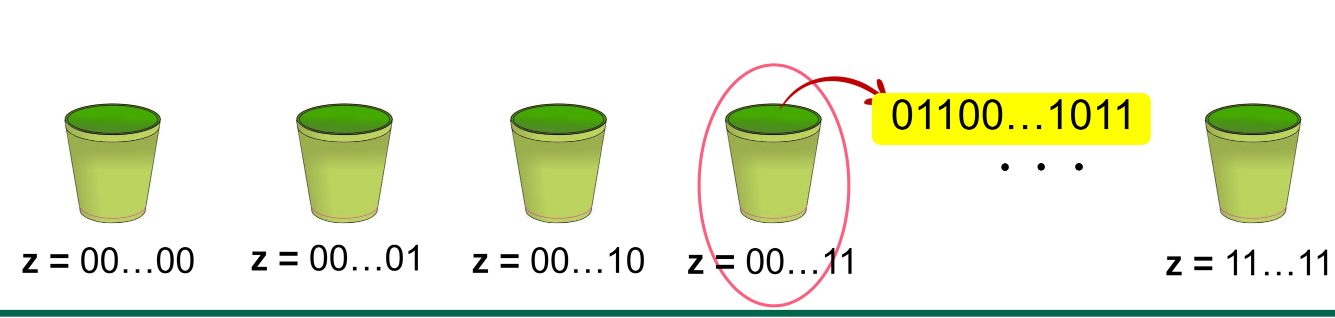
associated parity bits

LDPC code will be used as a hash function for the typical sequences!

Idea: reverse the process to do sampling!

LDPC-Based Sampling Scheme





1. Construct the combined graph
2. Flip K independent fair coins to determine the parity bits
3. Find the most likely sequence corresponds to the selected parity bits.

Theoretical Correctness

Theorem 1: Given $P_{\boldsymbol{x}}(\cdot)$, $\epsilon > 0, \delta > 0$, t large enough s.t. $\mathbb{P}(\mathcal{T}_\epsilon^t) \ge 1 - \frac{\delta}{2}$. Then there exists an LDPC code with parity matrix $\mathbb{H} \in \{0,1\}^{K \times Nt}$ where $K \sim NtH_0$, for which the proposed sampling scheme produces a sequence in $\mathcal{T}_\epsilon^t$ with probability at least $1 - \delta$.

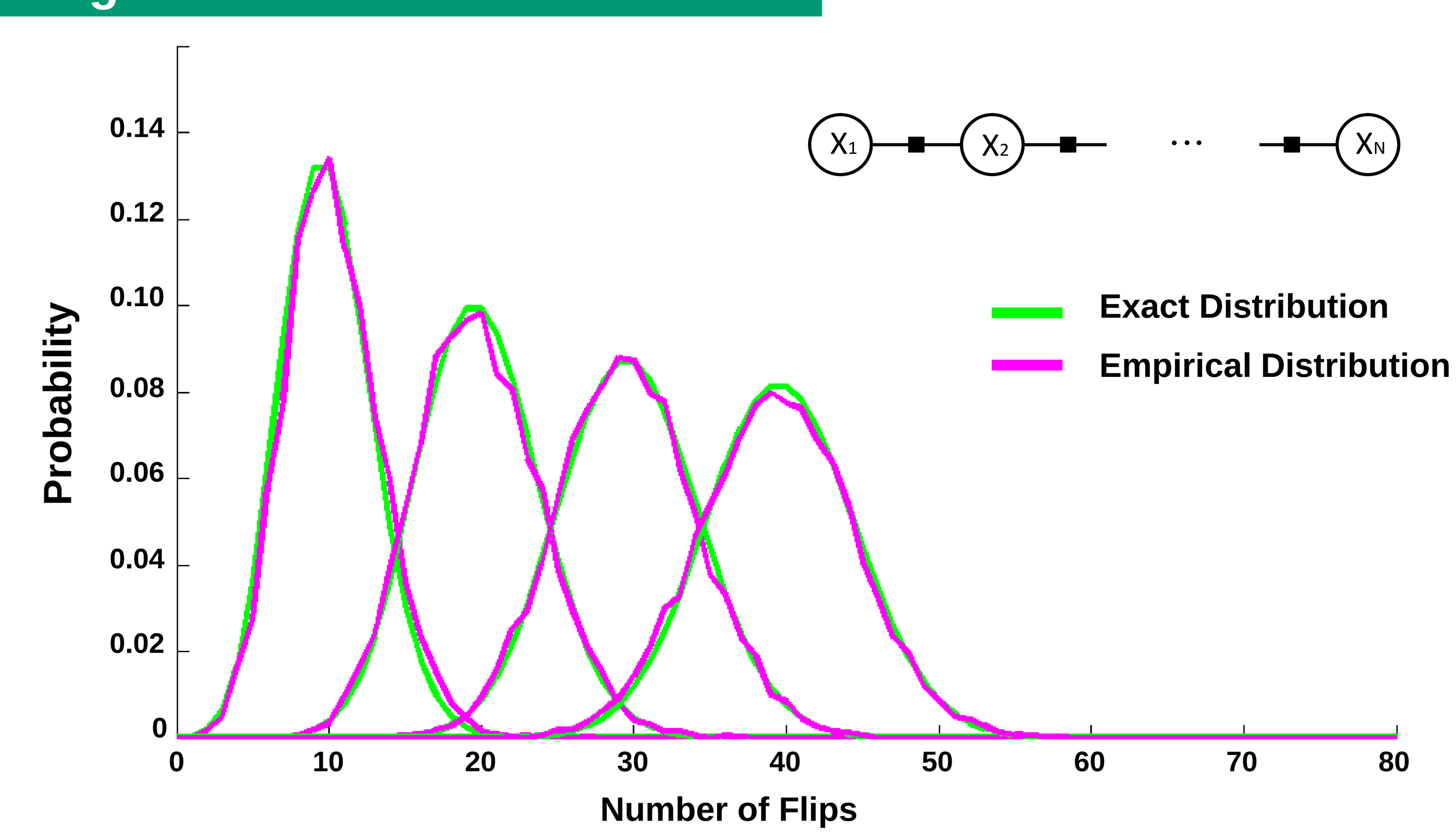
Theorem 2: If $H_0 < 1$, it is sufficient to have

$$t > \frac{1}{H_0 + \epsilon - \log_2(1 + H_0 \ln 2 + o\left(\frac{1}{N}\right))} \log_2\frac{4}{\delta} \approx \frac{2}{2\epsilon + H_0^2 \ln 2}\log_2\frac{4}{\delta}$$

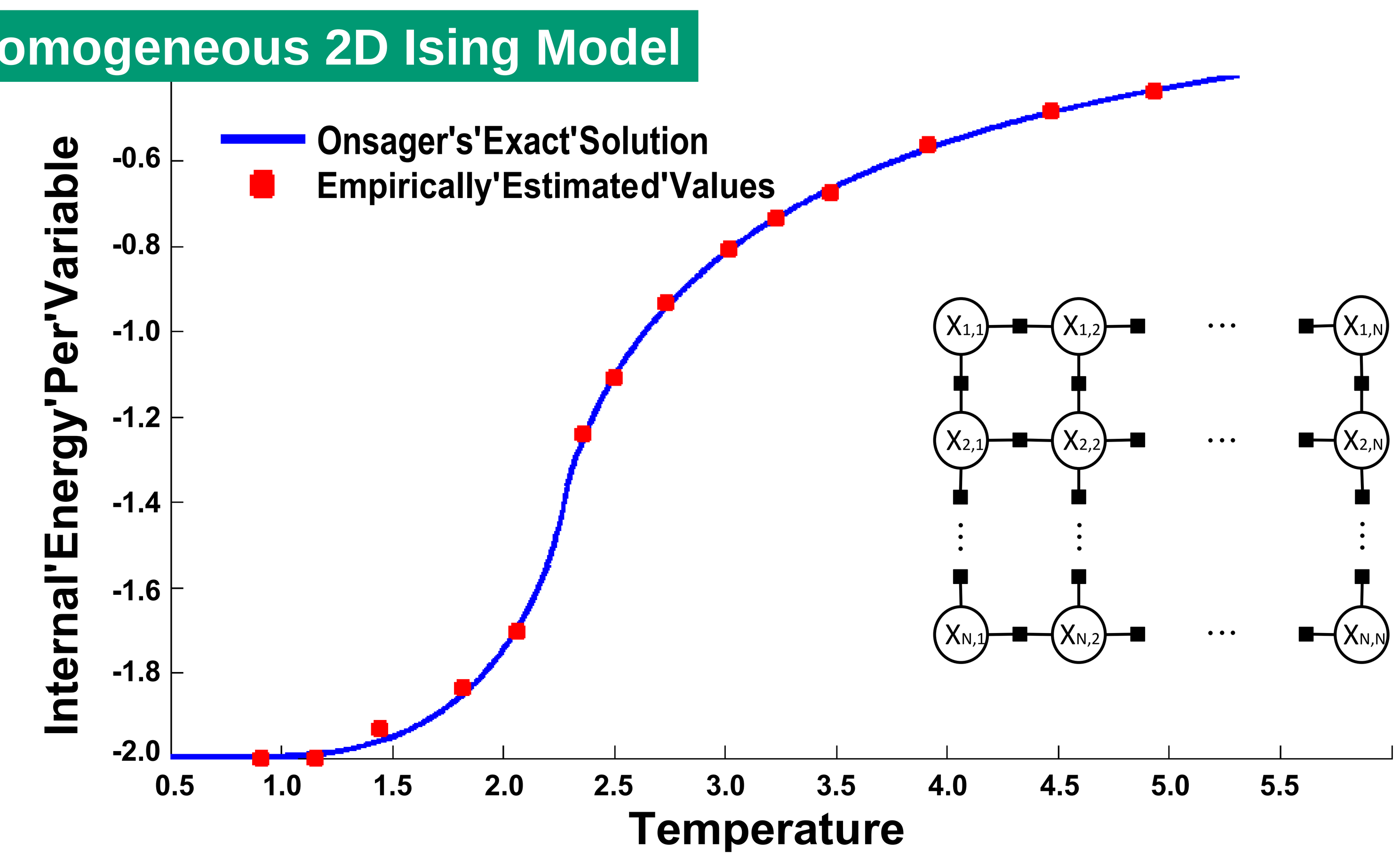
Experiment Results

In practice: step 3 infeasible ⇒ **Belief Propagation**

Homogeneous Markov Chain Model



Homogeneous 2D Ising Model



Discussion

The 'best' sampling method is application-dependent: different methods provide different trade-offs.

	Rejection Sampling	Importance Sampling	MCMC Methods	Proposed Scheme
Applicable to Continuous Variables?	Yes	Yes	Yes	No
Quality of Proposal Distribution Vital?	Yes	Yes	No	No
Applicable to High-Dimension Data?	No	No	Yes	Yes
Guaranteed Correctness?	Yes	No	Asymptotic	Approximate
Independent Samples?	Yes	N/A	No	Approximate

Highlights of desirable features of the proposed scheme:

a. **Good scaling behaviour** due to distributed nature of message-passing.
b. **Economical use of randomness**: close to the theoretical minimum.
c. **Good control over independence properties**