The Variational Coupled Gaussian Process Dynamic Model
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Introduction
Planning and execution of full-body movements is a hard control problem. Modular movement primitives (MPs) have been suggested as a means to simplify it while retaining a sufficient degree of control flexibility for a wide range of tasks. A particularly well-developed type of MP in robotics is the dynamical movement primitive (DMP) [Schaal, 2006]. But the form of the differential equation of a DMP remains fixed during learning, potentially reducing the representational capacity. Coupled Gaussian Process Dynamical Model (CGPDM) [Hoffman et al., 2016] combines the advantages of modularity and flexibility in the dynamics, at least theoretically. Non-parametric GP's learning has high computational cost. We improve this by employing sparse variational approximations and obtaining the need for sampling.

Coupled GPDMs model
Full dynamical system comprises M parts where every part is a modular MP (here 2 for simplicity). We introduce M × M dynamics in the latent space, each of which makes prediction about a part of the latent space from the previous state of some (other) part, these predictions are combined via PoE:

\[
\sigma^2 = \left( \sum_{m} \sigma^2_{m} \right)^{-1}
\]

\[
p(\mathbf{x} | \mathbf{u}; \sigma) = \prod_{m} N(\mathbf{x} | \mu_{m}, \Sigma_{m})
\]

Integrate out the intermediate predictions and get a simplified model:

\[
\text{Covariance matrix of latent points } x:
\]

\[
\Sigma = \text{PKPR}^{-1} \Sigma = \Sigma \Sigma_{\text{approx}} + \Sigma_{\text{res}} + \Sigma_{\text{noise}}
\]

Kernel function \( k_{\sigma} \) generates these covariances:

\[
\sigma^2_{k}(x, x') = \sigma^2 \left( \exp \left( \frac{-|x-x'|^2}{\sigma^2} \right) \right)
\]

\( \sigma_{\text{noise}} \) can be increased to control the coupling. Coupling strength:

\[
\sigma^2_{\text{noise}}(x, x') = \left( \frac{\sigma^2}{\sigma^2_{\text{noise}}} \right)
\]

Augmented model
Y - observed data
X - latent points
\( r_{t} \rightarrow v_{t} \) and \( z_{t} \rightarrow u_{t} \) - augmenting mappings

\[
\begin{align*}
X &= \{x_{1}, \ldots, x_{T}\}, x_{t} \in \mathbb{R}^{D} \\
Y &= \{y_{1}, \ldots, y_{T}\}, y_{t} \in \mathbb{R}^{D} \\
f(x_{t}) &\sim \mathcal{GP}(0, k_{\sigma}(x_{t}, x_{t}')) \\
f_{1} &= f(x_{t}) \\
x_{t+1} &\sim N(f_{t+1}, \Sigma_{t+1}) \\
g_{s}(X) &\sim \mathcal{GP}(0, k_{\sigma}(x_{t}, z_{t})) \\
g &= g(X) \\
Y_{d} &\sim N(g_{d}(I), \Sigma_{d}) \\
p(x_{0}) &= N(x_{0}|0, I)
\end{align*}
\]

For each part:

\[
\begin{align*}
X &= \{x_{1}, \ldots, x_{T}\}, x_{t} \in \mathbb{R}^{D} \\
Y &= \{y_{1}, \ldots, y_{T}\}, y_{t} \in \mathbb{R}^{D} \\
f(x_{t}) &\sim \mathcal{GP}(0, k_{\sigma}(x_{t}, x_{t}')) \\
f_{1} &= f(x_{t}) \\
x_{t+1} &\sim N(f_{t+1}, \Sigma_{t+1}) \\
g_{s}(X) &\sim \mathcal{GP}(0, k_{\sigma}(x_{t}, z_{t})) \\
g &= g(X) \\
Y_{d} &\sim N(g_{d}(I), \Sigma_{d}) \\
p(x_{0}) &= N(x_{0}|0, I)
\end{align*}
\]

For Y:

\[
\begin{align*}
\mathcal{L}_{\text{ELBO}} &= \mathbb{E}_{q(\theta)} \left[ \log p(Y | X, \theta) \right] + \text{KL}(q(\theta) || \pi(\theta))
\end{align*}
\]

For X:

\[
\begin{align*}
\mathcal{L}_{\text{ELBO}} &= \mathbb{E}_{q(\theta)} \left[ \log p(X | \theta) \right] + \text{KL}(q(\theta) || \pi(\theta))
\end{align*}
\]

Variational learning
Proposal variational distributions: \( q(z_{t}) = N(\mu_{z_{t}}, \Sigma_{z_{t}}) \), \( q(u) \) and \( q(v) \) are unconstrained distributions. The full variational joint proposal distribution is given:

\[
q(x, u, f, v, g) = q(y | g) q(x | g) q(v | x) q(u | x) q(f | u, v, g)
\]

The ELBO:

\[
\mathcal{L}(\theta) = \mathbb{E}_{q(\theta)} \left[ \log p(Y | X, \theta) \right] + \text{KL}(q(\theta) || \pi(\theta))
\]

Separating the latent dynamics part and applying the sufficient statistics assumption \( q(x, u) = q(u | x) \prod_{t \geq 1} q(f_{t} | x_{t-1}, u_{t}) \) [Hoffman et al., 2016], we get even much lower bound:

\[
\mathcal{L}(\theta) \geq \sum_{t = 1}^{T} \mathbb{E}_{q(\theta)} \left[ \log p(g_{s}(X_{t}), f_{t} | x_{t-1}, u_{t}) \right] + \mathbb{E}_{q(\theta)} \left[ \log p(x_{t}, u_{t}, f_{t} | x_{t-1}, u_{t}) \right] d\mu_{x_{t-1}} d\mu_{u_{t}} d\mu_{f_{t}}
\]

where \( H(x | q) \) is the entropy of \( q(x) \). The first integral is given in [Hoffman et al., 2016]. The remaining integral is derived below.

\[
A = \mathbb{E}_{q(\theta)} \left[ \log p(x_{t}, u_{t}, f_{t} | x_{t-1}, u_{t}) \right] d\mu_{x_{t-1}} d\mu_{u_{t}} d\mu_{f_{t}}
\]

\[
= \mathbb{E}_{q(\theta)} \left[ \log p(x_{t}, u_{t}, f_{t} | x_{t-1}, u_{t}) \right] d\mu_{x_{t-1}} d\mu_{u_{t}} d\mu_{f_{t}}
\]

The integrals over \( d\mu_{x_{t-1}} d\mu_{u_{t}} d\mu_{f_{t}}
\]

\[
B = \mathbb{E}_{q(\theta)} \left[ \log p(x_{t}, u_{t}, f_{t} | x_{t-1}, u_{t}) \right] d\mu_{x_{t-1}} d\mu_{u_{t}} d\mu_{f_{t}}
\]

Future work
- Check the advantage of the variational learning vs. MAP
- Train on real MoCap datasets
- Sensory-motor integration as interaction in latent space dynamics
- Deep models

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References


Toy dataset
ARD RBF kernel, 2-nd order dynamics, 4 inducing inputs for dynamics mapping (circles), 8 inducing inputs for \( Y \rightarrow X \) mapping (pluses)